

## The spectral action for Moyal planes

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### Abstract

Extending a result of D. V. Vassilevich [49], we obtain the asymptotic expansion for the trace of a spatially regularized heat operator  $L^\Theta(f)e^{-t\Delta^\Theta}$ , where  $\Delta^\Theta$  is a generalized Laplacian defined with Moyal products and  $L^\Theta(f)$  is Moyal left multiplication. The Moyal planes corresponding to any skewsymmetric matrix  $\Theta$  being spectral triples [24], the spectral action introduced in noncommutative geometry by A. Chamseddine and A. Connes [6] is computed. This result generalizes the Connes-Lott action [15] previously computed by Gayral [23] for symplectic  $\Theta$ .

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# 1 Introduction

Since few years, the interest in noncommutative field theory has been renewed in many works. The noncommutative space is quite often of Moyal type, involving either noncommutative tori or Moyal planes (see [19, 33] for recent reviews).

Historically, Moyal [39] has tried to build quantum mechanics with a statistical point of view using a phase space approach. Actually, his idea was that the formalism of quantum theory allows to derive the phase space distributions  $F(p, q)$  when a theory of functions of noncommuting observables is specified and conversely. This type of consideration was initiated by Wigner [53] with a formula for  $F(p, q)$  using Fourier transform and for canonical conjugate coordinates and momenta by Weyl [51] with group theoretical motivations. A noncommutative star product was previously explicitly given by Groenewold [29]. In fact, the use of quantification by deformation [1, 42] has been intensively investigated since it yields a continuous path between classical and quantum mechanics. In the meantime, the Weyl–Wigner quantification process was also interesting for the pseudodifferential operators theory [22]. The Seiberg–Witten [44] map allows to go from ordinary to noncommutative gauge field theory and the replacement of the ordinary commutative product of functions by the Moyal noncommutative one is now ubiquitous in string theory where the effective low energy theory of D-branes with B-field background lives in a noncommutative space.

The mathematical background of these different developments for quantization within noncommutative geometry [9, 28] where noncommutative tori [7, 41] play an important role [12], includes the construction of new spectral triples [3, 13, 14, 16, 24], and more generally the theory of pseudodifferential operators [18, 45], the construction of star product [26, 34], integrable systems etc. For reviews on these topics, see [9, 28, 35, 37, 38].

It has been proposed for a long time [46, 54] that the noncommutative spacetime is a quantum effect of gravity and that this could provide some hints for the regularization of quantum field theory. Naturally, many types of action have been proposed and here we choose the spectral action introduced by Chamseddine and Connes [5, 6] in a proper noncommutative geometry setting. Since a similar action was derived in [23, 24] following a prescription by Connes and Lott [15], it is interesting to quote the differences. In [6], the idea is to recover the usual action of the standard model of particle physics from purely gravitational considerations; more precisely to define the bosonic action by  $\text{Tr}(\phi(\mathcal{D}^2/\Lambda^2))$  while the fermionic action is simply  $\langle \psi, \mathcal{D}\psi \rangle$  where  $\mathcal{D}$  is the Dirac operator,  $\Lambda$  is an energy scale cut-off and  $\phi$  is a smooth positive function. So Chamseddine and Connes recovered the Einstein plus Yang–Mills and Weyl actions including of course the spin 1 bosons, but also the part induced by the Higgs fields of spin 0. The action functional computed in [23, 24] is defined by  $\text{Tr}^+(F^2\mathcal{D}^{-d})$  where  $F$  is the field strength curvature of a vector potential and  $\text{Tr}^+$  is a Dixmier trace which pins down the leading term of logarithmic divergence in the usual trace of  $F^2$ .

Here we choose for the manifold the Moyal plane  $\mathbb{R}^{2m}$  with flat curvature, technically viewed as an algebra  $(\mathcal{S}(\mathbb{R}^{2m}), \star_\theta)$  for the Moyal product  $\star_\theta$  ( $\mathcal{S}$  is the Schwartz space) and as Dirac operator, the usual one  $-i(\partial_\mu + L^\Theta(\omega_\mu)) \otimes \gamma^\mu$  but where the connection  $\omega$  acts by Moyal left multiplication  $L^\Theta(\omega)$  on the usual Hilbert space of  $L^2$ -sections of the trivial spinor bundle spinors  $\mathcal{H} = L^2(\mathbb{R}^{2m}) \otimes \mathbb{C}^{2^m}$ . In [24], this algebra and one of its unitizations is proved to be a real spectral triple of spectral dimension  $2m$ . So this case completely fits the requirements and the computation of  $\text{Tr}(\phi(\mathcal{D}/\Lambda))$  is possible using a standard heat kernel technique. The not so surprisingly result is that one recovers the usual commutative action where all commutative products have been replaced by the Moyal ones.

Despite the fact that this computation is made here in Euclidean signature and with no real gravity, the main interest is that all the algebraic and analytical difficulties are overcome and

that it is the first example of spectral action for a not almost commutative spectral triple. Since the Dirac operator has a noncompact resolvent, a spatial regularization by the multiplication of a function  $f$  in the algebra is introduced to get the tracability of  $L^\Theta(f)e^{-t\Delta^\Theta}$  where  $\Delta^\Theta$  is the generalized Moyal Laplacian. Of course, the choice of a regularization is arbitrary and one could prefer for instance a soliton one [36], when one wants to avoid, before the limit, the renormalization problems set in by UV/IR mixing.

After some reminders on the role of the Moyalology for spectral triples in section 2, we establish the main result, and then, give the important technical details on the heat kernel computation in section 3. Section 4 is just an application to the case of Moyal planes which ends with few remarks on the difficulties with non flat cases.

## 2 Moyal spectral triples and spectral action

### 2.1 Moyal analysis

In this section, the very basic tools on Moyal analysis are recalled and we refer to [24] for a review. The Moyal product comes from the phase space formulation of flat quantum mechanics, that is a deformation of the associative algebra structure of a suitable family of functions on  $\mathbb{R}^{2m}$  with pointwise product in the direction of the flat Poisson bracket  $\{.,.\}_P$ . More precisely, if we denote by  $W$  the Weyl map which assigns Schwartz functions on  $\mathbb{R}^{2m}$  say, to bounded operators on  $L^2(\mathbb{R}^m)$ , the Moyal product  $\star_\theta$  is constructed in order to obtain a  $*$ -algebra homomorphism

$$W : \mathcal{S}(\mathbb{R}^{2m}) \rightarrow \mathcal{L}(L^2(\mathbb{R}^m)), \quad W(f \star_\theta g) = W(f) W(g).$$

This leads us to

$$(f \star_\theta g)(x) := (\pi\theta)^{-2m} \iint_{\mathbb{R}^{2m} \times \mathbb{R}^{2m}} f(y) g(z) e^{\frac{2i}{\theta}(x-y) \cdot S(x-z)} d^{2m}y d^{2m}z, \quad (1)$$

where  $S := \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$  comes from the canonical symplectic structure of  $\mathbb{R}^{2m} = T^*\mathbb{R}^m$  and  $\theta \in \mathbb{R}_+^*$  is the deformation parameter.

Actually, one can define Moyal products on  $\mathcal{S}(\mathbb{R}^n)$ ,  $n$  even or odd, independently of any symplectic structure. In those cases, it is any real skewsymmetric matrix  $\Theta$  which defines the deformation directions. Generic Moyal products are then defined by

$$(f \star_\Theta g)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\xi(x-y)} f(x - \frac{1}{2}\Theta\xi) g(y) d^n y d^n \xi. \quad (2)$$

To fix notations and avoid to refer to the odd or even case,  $n$  will be an integer equals to the plane dimension and  $m$  will be the integer part of  $\frac{n}{2}$ .

This formula shows that the theory of pseudodifferential operators on  $\mathbb{R}^n$  [31, 45] is suitable for the analysis of left and right Moyal multiplication operators  $L^\Theta(f)$  and  $R^\Theta(f)$  defined by  $L^\Theta(f)\psi := f \star_\Theta \psi$  and  $R^\Theta(f)\psi := \psi \star_\Theta f$ . In particular their symbols are

$$\sigma[L^\Theta(f)](\xi, x) = f(x - \frac{1}{2}\Theta\xi), \quad \sigma[R^\Theta(f)](\xi, x) = f(x + \frac{1}{2}\Theta\xi). \quad (3)$$

On coordinate functions  $x^\mu$ ,  $\mu = 1, \dots, n$ , generic Moyal products formally define a generalized Heisenberg Lie algebra structure:

$$[x^\mu, x^\nu]_{\star_\Theta} = i\Theta^{\mu\nu} 1.$$

This equality can have a precise analytical meaning if we work with Moyal products on some tempered distribution spaces [27, 47], and is obvious from the asymptotic expansion of Moyal products:

$$f \star_{\Theta} g \sim \sum_{\alpha \in \mathbb{N}^n} \left(\frac{i}{2}\right)^{|\alpha|} \frac{1}{\alpha!} \frac{\partial f}{\partial x^{\alpha}} \frac{\partial g}{\partial (\Theta x)^{\alpha}}. \quad (4)$$

This expansion can be heuristically derived from (2) by a Taylor expansion of  $\sigma[L^{\Theta}(f)](\xi, x)$  "near"  $x$ , for a more rigorous approach see [20]. Moyal products satisfy a few useful algebraic equalities (see [24] for a review); in particular the Leibniz rule is satisfied, the integral is a faithful trace and the complex conjugation is an involution:

$$\partial^{\mu}(f \star_{\Theta} g) = \partial^{\mu} f \star_{\Theta} g + f \star_{\Theta} \partial^{\mu} g, \quad (5)$$

$$\int (f \star_{\Theta} g)(x) d^n x = \int f(x) g(x) d^n x, \quad (6)$$

$$(f \star_{\Theta} g)^* = g^* \star_{\Theta} f^*. \quad (7)$$

These properties allow to prove that  $\mathcal{B}_{\Theta} := (\mathcal{S}(\mathbb{R}^n), \star_{\Theta})$  is an associative and involutive Fréchet algebra with a jointly continuous product.

For  $\Theta$  symplectic, hence  $n = 2m$ , it is proved in [27, 47] that  $(L^2(\mathbb{R}^{2m}), \star_{\theta})$  is an associative Banach algebra, and we have shown in [24] that  $(\mathcal{D}_{L^2}(\mathbb{R}^{2m}), \star_{\theta})$  is also an  $*$ -algebra with a jointly continuous product,  $\mathcal{D}_{L^2}(\mathbb{R}^{2m})$  being the space of smooth functions, having all their derivatives in  $L^2(\mathbb{R}^{2m})$  endowed with the Fréchet topology of  $L^2$ -convergence for all derivatives. For the nonunital spectral triple point of view (see below), one needs also to choose a unitization for these algebras. In [24] is studied the unital  $*$ -algebra  $(\mathcal{O}_0(\mathbb{R}^{2m}), \star_{\theta})$ , where  $\mathcal{O}_0(\mathbb{R}^{2m})$  consists in smooth bounded functions with bounded derivatives, with the topology given by the family of semi-norms  $\{p_{\alpha}\}_{\alpha \in \mathbb{N}^{2N}}$ ,  $p_{\alpha}(f) := \|\partial^{\alpha} f\|_{\infty}$ .

For generic Moyal products,  $(\mathcal{S}(\mathbb{R}^n), \star_{\Theta})$ ,  $(\mathcal{D}_{L^2}(\mathbb{R}^n), \star_{\Theta})$ ,  $(\mathcal{O}_0(\mathbb{R}^n), \star_{\Theta})$  are also Fréchet algebras. Actually this statement comes from the algebra structure of  $\mathcal{S}(\mathbb{R}^n)$ ,  $\mathcal{D}_{L^2}(\mathbb{R}^n)$ ,  $\mathcal{O}_0(\mathbb{R}^n)$  with pointwise product as well as with Moyal one, and because any Moyal product splits into a symplectic Moyal product and a pointwise one denoted by a point [42, Proposition 2.7 and Corollary 2.8]: Namely,  $f \star_{\Theta} g(x) = f(x) \cdot g(x)$  when  $\Theta = 0$ , so if the matrix  $\Theta$  is decomposed as the direct sum of a symplectic one  $\theta$  of dimension  $2m$  and the zero matrix of dimension  $n - 2m$ , then  $(\mathcal{S}(\mathbb{R}^n), \star_{\Theta}) \cong (\mathcal{S}(\mathbb{R}^{2m}), \star_{\theta}) \hat{\otimes} (\mathcal{S}(\mathbb{R}^{n-2m}), \cdot)$ . Remark in particular that  $\mathcal{S}(\mathbb{R}^n)$  and  $\mathcal{O}_0(\mathbb{R}^n)$  are algebras for pointwise product while for  $\mathcal{D}_{L^2}(\mathbb{R}^n)$  this is a consequence of the inclusion  $\mathcal{D}_{L^2}(\mathbb{R}^n) \subset \mathcal{O}_0(\mathbb{R}^n)$  [43].

A *spectral triple*  $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{H}, \mathcal{D}, J, \chi)$  (a noncommutative generalization of a Riemannian spin manifold) consists of an algebra  $\mathcal{A}$ , a suitable one of its unitizations  $\tilde{\mathcal{A}} \supset \mathcal{A}$  (for the analogue of the noncompact case) both faithfully represented by bounded operators on a separable Hilbert space  $\mathcal{H}$  (the representation is denoted by  $\pi$ ), together with an unbounded selfadjoint operator  $\mathcal{D}$  such that  $\pi(a)(\mathcal{D} + \lambda)^{-1}$  is a compact operator for all  $a \in \mathcal{A}$  and  $\lambda$  in the resolvent of  $\mathcal{D}$  and such that the commutators  $[\mathcal{D}, \pi(a)]$  for all  $a \in \tilde{\mathcal{A}}$  extend to bounded operators.  $J$  and  $\chi$  are respectively antiunitary and unitary operators with commutation relations depending on the dimension of the triple. These data must moreover fulfill a set of axioms (see [10, 24]).

It is shown in [24] that symplectic Moyal planes yield nonunital spectral triples if we choose  $\mathcal{A} = (\mathcal{D}_{L^2}(\mathbb{R}^{2m}), \star_{\theta})$ ,  $\tilde{\mathcal{A}} = (\mathcal{O}_0(\mathbb{R}^{2m}), \star_{\theta})$ , represented by the diagonal left regular representation  $\pi^{\theta}(f) := L^{\theta}(f) \otimes 1_{2m}$  on the Hilbert space of  $L^2$ -sections of the trivial spinor bundle  $\mathcal{H} = L^2(\mathbb{R}^{2m}) \otimes \mathbb{C}^{2^N}$ , and for  $\mathcal{D}$ , the flat Dirac operator  $\not{D} := -i\partial_{\mu} \otimes \gamma^{\mu}$  where  $\gamma^{\mu}$  are the Clifford matrices associated to  $(\mathbb{R}^{2m}, \eta)$  with  $\eta$  the standard Euclidean metric of  $\mathbb{R}^{2m}$ .

## 2.2 Main result

The *action functional* or Connes–Lott action [15] associated with this spectral triple gives the noncommutative Yang–Mills action for symplectic Moyal products:

$$YM(\alpha) = c \int F^{\mu\nu} \star_{\theta} F_{\mu\nu} d^{2m}x, \quad (8)$$

where  $\alpha$  is a universal represented connection,

$$\alpha \in \tilde{\pi}^{\theta}(\{a_0 \delta a_1 : a_0, a_1 \in \mathcal{B}_{\theta}\}) = \left\{ \pi^{\theta}(a_0)[\not\partial, \pi^{\theta}(a_1)] : a_0, a_1 \in \mathcal{B}_{\theta} \right\},$$

and  $F$  is its curvature:  $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} + [A^{\mu}, A^{\nu}]_{\star_{\theta}}$  and  $A^{\mu}$  being defined by  $\alpha = L^{\theta}(A_{\mu}) \otimes \gamma^{\mu}$ . This result comes from the Junk computation [23] and the following result [24]:

**Theorem 2.1.** *For  $f \in \mathcal{S}(\mathbb{R}^{2m})$ , the compact operator  $\pi^{\theta}(f)(\not{D}^2 + \varepsilon^2)^{-m}$  lies in  $\mathcal{L}^{(1,\infty)}(\mathcal{H})$  and any of its Dixmier traces  $Tr_{\omega}$  is independent of the positive number  $\varepsilon$ . More precisely we have,*

$$Tr_{\omega}(\pi^{\theta}(f)(\not{D}^2 + \varepsilon^2)^{-m}) = \frac{2^m \Omega_{2m}}{2m(2\pi)^{2m}} \int f(x) d^{2m}x, \quad (9)$$

where  $\mathcal{L}^{(1,\infty)}(\mathcal{H})$  is the ideal of compact operators whose  $k$ -th singular values satisfy  $\mu_k(T) = O(k^{-1})$  and  $\Omega_{2m}$  is the hyper-area of the unit sphere in  $\mathbb{R}^{2m}$ .

For generic Moyal products,  $((\mathcal{D}_{L^2(\mathbb{R}^n)}, \star_{\theta}), (\mathcal{O}_0(\mathbb{R}^n), \star_{\theta}), L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2m}, \not\partial)$  yields also a nonunital spectral triple, but the Connes–Lott action computation is not obvious because the computation of (8) was basis dependent.

Let  $\Delta^{\Theta}$  be a *noncommutative generalized Laplacian* associated with Moyal products

$$\begin{aligned} \Delta^{\Theta} &:= -\left(\eta^{\mu\nu}(\partial_{\mu} + L^{\Theta}(\omega_{\mu}))(\partial_{\nu} + L^{\Theta}(\omega_{\nu})) + L^{\Theta}(E)\right) \otimes 1_{2m}, \\ \Delta^{\Theta} &=: \Delta_r^{\Theta} \otimes 1_{2m} \end{aligned} \quad (10)$$

acting on the Hilbert space  $\mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2m} =: \mathcal{H}_r \otimes \mathbb{C}^{2m}$ , where  $\omega_{\mu}^* = -\omega_{\mu}$  and  $E$  are in  $\mathcal{O}_0(\mathbb{R}^n)$ . From now, let  $\mathcal{B}_{\Theta} := (\mathcal{S}(\mathbb{R}^n), \star_{\Theta})$  acting on  $\mathcal{H}$  by the diagonal left regular representation  $\pi^{\Theta}(\cdot) := L^{\Theta}(\cdot) \otimes 1_{2m}$ .

For  $f \in \mathcal{B}_{\Theta}$ ,  $L^{\Theta}(f)e^{-t\Delta^{\Theta}}$  will be called *spatially regularized heat operator* associated with the generalized Laplacian  $\Delta^{\Theta}$ .

The following is the main result.

**Theorem 2.2.** *Let  $\Delta^{\Theta}$  be as in (10) and  $f \in \mathcal{B}_{\Theta}$ . Then  $Tr\left(\pi^{\Theta}(f)e^{-t\Delta^{\Theta}}\right)$  has an asymptotic expansion*

$$Tr\left(\pi^{\Theta}(f)e^{-t\Delta^{\Theta}}\right) \sim_{t \rightarrow 0} 2^m \left(\frac{1}{4\pi t}\right)^{n/2} \sum_{l \in \mathbb{N}} t^l \int_{\mathbb{R}^n} f(x) \tilde{a}_{2l}(x) d^n x, \quad (11)$$

where

$$\begin{aligned} \tilde{a}_0(x) &= 1, \\ \tilde{a}_2(x) &= E(x), \\ \tilde{a}_4(x) &= \frac{1}{2} E \star_{\Theta} E(x) + \frac{1}{6} \eta^{\mu\nu} E_{;\mu\nu}(x) + \frac{1}{12} \Omega^{\mu\nu} \star_{\Theta} \Omega_{\mu\nu}(x), \\ \tilde{a}_6(x) &= \frac{1}{6} E \star_{\Theta} E \star_{\Theta} E(x) + \frac{1}{12} \eta^{\mu\nu} E_{;\mu} \star_{\Theta} E_{;\nu}(x) + \frac{1}{6} \eta^{\mu\nu} E \star_{\Theta} E_{;\mu\nu}(x) \\ &\quad + \frac{1}{60} \eta^{\mu\nu} \eta^{\rho\sigma} E_{;\mu\nu\rho\sigma}(x) + \frac{1}{12} E \star_{\Theta} \Omega^{\mu\nu} \star_{\Theta} \Omega_{\mu\nu}(x) + \frac{1}{45} \eta^{\rho\sigma} \Omega^{\mu\nu}_{;\rho} \star_{\Theta} \Omega_{\mu\nu;\sigma}(x) \\ &\quad + \frac{1}{180} \eta^{\rho\sigma} \Omega^{\mu\nu}_{;\nu} \star_{\Theta} \Omega_{\mu\rho;\sigma}(x) + \frac{1}{30} \eta^{\rho\sigma} \Omega^{\mu\nu} \star_{\Theta} \Omega_{\mu\nu;\rho\sigma} - \frac{1}{30} \Omega^{\mu\nu} \star_{\Theta} \Omega_{\nu\rho} \star_{\Theta} \Omega^{\rho}_{\mu}(x), \end{aligned}$$

where  $g_{;\mu} := \partial_{\mu} g + [\omega_{\mu}, g]_{\star_{\Theta}}$  and  $\Omega_{\mu\nu} := \partial_{\mu} \omega_{\nu} - \partial_{\nu} \omega_{\mu} + [\omega_{\mu}, \omega_{\nu}]_{\star_{\Theta}}$  is the curvature of the connection  $\omega$ .

### 3 Heat kernel expansion for Moyal generalized Laplacians

We will first discuss the heat kernel expansion for Laplace type operators associated with Moyal products, for NC planes as well as for NC tori. This section generalizes Vassilevich's result [49] in two directions: first the Moyal products are defined by their integral form as opposed to differential or formal Moyal products (4) and second they are taken over the whole plane  $\mathbb{R}^n$  and not only on NC tori. This noncompact situation generates some analytical difficulties.

We use the standard one-parameter semigroup theory of  $e^{-tA}$  where  $A$  is a positive (unbounded) operator and  $t \in \mathbb{R}^+$ . Let  $\mathcal{H}$  be a separable Hilbert space. We denote by  $\mathcal{B}(\mathcal{H})$  the set of bounded operators on  $\mathcal{H}$ , by  $\mathcal{K}(\mathcal{H})$  the compact one's and by  $\mathcal{L}^p(\mathcal{H})$  the  $p$ -th Schatten class.

If we assume that  $A$  is a nonnegative selfadjoint operator on  $\mathcal{H}$ , then  $e^{-zA}$  is holomorphic for  $\Re(z) > 0$  and  $\|e^{-zA}\| \leq 1$  [32, Example 1.25, p. 493] [55]. With  $\mathcal{R}_A(z) := (z - A)^{-1}$  denoting the resolvent of  $A$ , one can use the holomorphic functional calculus:

$$e^{-tA} = \frac{1}{2i\pi} \int_{\Gamma} e^{-tz} \mathcal{R}_A(z) dz, \quad (12)$$

where  $\Gamma$  is a positively oriented (possibly infinite) closed curve containing the spectrum of  $A$ .

**Lemma 3.1.** *Let  $B$  be a bounded operator and  $A$  be a nonnegative densely defined generator of a holomorphic semigroup such that  $B\mathcal{R}_A(z)^l \in \mathcal{L}^1(\mathcal{H})$  for some  $z \notin \text{Spect}(A)$ . Then for  $t > 0$ ,  $Be^{-tA}$  is trace-class.*

*Proof.* For some  $z_0 \notin \text{Spect}(A)$ , the semigroup property together with the first resolvent equation and (12) gives:

$$B e^{-tA} = B (e^{-\frac{t}{l}A})^l = B \mathcal{R}_A(z_0)^l \left( \frac{1}{2i\pi} \int_{\Gamma} e^{-\frac{t}{l}z} (1 + (z - z_0)\mathcal{R}_A(z)) dz \right)^l.$$

This concludes the proof because  $\|\mathcal{R}_A(z)\| \leq \frac{M}{|z|}$  for all  $z$  with  $\Re(z) > 0$ , thus

$$\int_{\Gamma} e^{-\frac{t}{l}\Re(z)} \left( 1 + |z - z_0| \|\mathcal{R}_A(z)\| \right) |dz| < \infty. \quad \square$$

Since we are interested in the small  $t$ -asymptotic expansion of  $\text{Tr}(Be^{-tA})$ , recall the following definition:

Let  $\{f_n\}_n$  be a sequence of functions such that  $f_n(t) \neq 0$  for  $t \neq 0$  and  $f_{n+1}(t) = o(f_n(t))$  as  $t \rightarrow 0$ . A function  $f$  has the *asymptotic expansion*  $f(t) \sim_{t \rightarrow 0} \sum_{n=0}^{\infty} a_n f_n(t)$ , when for each  $k \in \mathbb{N}$ ,  $f(t) = \sum_{n=0}^k a_n f_n(t) + O(f_{k+1}(t))$  as  $t \rightarrow 0$ .

#### 3.1 Heat kernel expansion for Moyal planes

We will first show that  $L^{\Theta}(f)e^{-t\Delta_r^{\Theta}}$  is trace-class for  $t \in \mathbb{R}_+^*$ , then we will show that its trace has a small  $t$ -asymptotic expansion:

$$\text{Tr} \left( L^{\Theta}(f)e^{-t\Delta_r^{\Theta}} \right) \sim_{t \rightarrow 0} \left( \frac{1}{4\pi t} \right)^{n/2} \sum_{l \in \mathbb{N}} t^l \int_{\mathbb{R}^n} f(x) \tilde{a}_{2l}(x) d^n x,$$

where the local invariants  $\tilde{a}_l$  are built from the universal (represented) connection  $\omega_{\mu}$ , the (nonlocal) endomorphism  $E$  and their covariant derivative (in the adjoint representation)  $\partial_{\mu} + L^{\Theta}(\omega_{\mu}) - R^{\Theta}(\omega_{\mu})$ .

We will prove that  $L^{\Theta}(f)e^{-t\Delta_r^{\Theta}}$  is trace-class by two approaches. The first uses semigroup theory results while the second will be based on pseudodifferential operator ( $\Psi$ DO) techniques, which is more in the spirit of [24].

**Theorem 3.2.** *Let  $f \in \mathcal{S}(\mathbb{R}^n)$ ,  $\omega_\mu, E \in \mathcal{O}_0(\mathbb{R}^n)$  with  $\omega_\mu^* = -\omega_\mu$  and  $E = -g^* \star_\Theta g$  for some  $g \in \mathcal{O}_0(\mathbb{R}^n)$ . Then, for all  $t > 0$  the spatially regularized heat operator associated with  $\Delta^\Theta$  defined in (10) is trace-class.*

*First proof of Theorem 3.2.* Because  $(L^\Theta(g))^* = L^\Theta(g^*)$ ,  $\Im(g) = 0$  implies  $L^\Theta(g)$  is selfadjoint, so  $\Delta^\Theta$  is positive. Thanks to Lemma 3.1, it is enough to prove that  $L^\Theta(f)R_{\Delta_r^\Theta}(z)^l$  is trace-class for  $l > \frac{n}{2}$ .

Let us anticipate further notations to see that  $\Delta^\Theta$  is a squared covariant Dirac operator:

$$\begin{aligned}\Delta^\Theta &= \not{\partial}_\omega^2 - B, \\ \not{\partial}_\omega &:= -i(\partial_\mu + L^\Theta(\omega_\mu)) \otimes \gamma^\mu,\end{aligned}$$

and  $B := L^\Theta(E) \otimes 1_{2^m} - L^\Theta(\partial_\mu(\omega_\nu) - \omega_\mu \star_\Theta \omega_\nu) \otimes (\eta^{\mu\nu} 1_{2^m} - \gamma^\nu \gamma^\mu)$  is bounded.

Assume first that  $l = 1$ ,  $z = -1$ ,  $B = 0$ . Using the notations  $\pi^\Theta(\omega) := L^\Theta(\omega_\mu) \otimes \gamma^\mu$ ,  $\pi^\Theta(\not{\partial}(f)) := L^\Theta(\partial_\mu f) \otimes \gamma^\mu$  and the fact that all  $f \in \mathcal{S}(\mathbb{R}^n)$  factorizes as  $f = f_1 \star_\Theta f_2$ , for some  $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$  [24, Proposition 2.7], one gets:

$$\begin{aligned}\pi^\Theta(f)\mathcal{R}_{\Delta^\Theta}(-1) &= -\pi^\Theta(f)\frac{1}{\not{\partial}-i}\left(1-\pi^\Theta(\omega)\frac{1}{\not{\partial}_\omega-i}\right)\frac{1}{\not{\partial}_\omega+i} \\ &= -\pi^\Theta(f_1)\frac{1}{\not{\partial}-i}\pi^\Theta(f_2)\left(1-\pi^\Theta(\omega)\frac{1}{\not{\partial}_\omega-i}\right)\frac{1}{\not{\partial}_\omega+i} \\ &\quad -\pi^\Theta(f_1)\frac{1}{\not{\partial}-i}\pi^\Theta(\not{\partial}(f_2))\frac{1}{\not{\partial}-i}\left(1-\pi^\Theta(\omega)\frac{1}{\not{\partial}_\omega-i}\right)\frac{1}{\not{\partial}_\omega+i} \\ &= -\pi^\Theta(f_1)\frac{1}{\not{\partial}-i}\pi^\Theta(f_2)\frac{1}{\not{\partial}+i}\left(1-\pi^\Theta(\omega)\frac{1}{\not{\partial}_\omega+i}\right) \\ &\quad +\pi^\Theta(f_1)\frac{1}{\not{\partial}-i}\pi^\Theta(f_2)\star_\Theta\omega\frac{1}{\not{\partial}-i}\left(1-\pi^\Theta(\omega)\frac{1}{\not{\partial}_\omega-i}\right)\frac{1}{\not{\partial}_\omega+i} \\ &\quad -\pi^\Theta(f_1)\frac{1}{\not{\partial}-i}\pi^\Theta(\not{\partial}(f_2)\star_\Theta\omega)\frac{1}{\not{\partial}-i}\left(1-\pi^\Theta(\omega)\frac{1}{\not{\partial}_\omega-i}\right)\frac{1}{\not{\partial}_\omega+i}.\end{aligned}$$

By [24, Lemmata 4.5 and 4.14],  $\pi^\Theta(g)\mathcal{R}_{\not{\partial}}(i)\pi^\Theta(h)\mathcal{R}_{\not{\partial}}(i) \in \mathcal{L}^p(\mathcal{H})$  whenever  $g, h \in \mathcal{S}(\mathbb{R}^n)$  and  $p > \frac{n}{2}$ . The boundness of  $\pi^\Theta(\omega)$  and  $\mathcal{R}_{\not{\partial}_\omega}(z)$  then yields  $\pi^\Theta(f)\mathcal{R}_{\Delta^\Theta}(-1) \in \mathcal{L}^p(\mathcal{H})$  for the same  $p$ . For  $l \geq 1$ , repeat this algorithm using  $(A+C)^{-1} = A^{-1}(1-CA^{-1}(\cdots(1-C(A+C)^{-1})\cdots))$  up to order  $l$ .

The case with non-zero  $B$  is obtained using the same trick:

$$\pi^\Theta(f)\mathcal{R}_{\Delta^\Theta}(-1) = \pi^\Theta(f)\mathcal{R}_{\not{\partial}_\omega^2}(-1)(1-B\mathcal{R}_{\Delta^\Theta}(-1)).$$

The first resolvent equation implies the same result for any  $z$  instead of  $-1$  in the resolvent set of  $\text{Spect}(\Delta^\Theta)$ .  $\square$

The second proof, which is based on a functional calculus for  $\Psi$ DO [18], needs the following definition of  $\Psi$ DO classes relevant for Moyal analysis (Shubin [45] or GLS [30] symbol classes).

**Definition 3.3.** Let  $S^{\rho,\lambda}$  be the Shubin or GLS symbol class

$$\begin{aligned}S^{\rho,\lambda} &:= \left\{ \sigma \in \mathcal{C}^\infty(\mathbb{R}^{2n}) : \forall \alpha, \beta \in \mathbb{N}^n, \exists C_{\alpha\beta} \in \mathbb{R}^+ \right. \\ &\quad \left. \left| \partial_x^\alpha \partial_\xi^\beta \sigma(\xi, x) \right| \leq C_{\alpha,\beta} (1+|x|^2)^{(\rho-|\alpha|)/2} (1+|\xi|^2)^{(\lambda-|\beta|)/2} \right\},\end{aligned}$$

and let  $\Psi^{\rho,\lambda} := \{A \in \Psi DO : \sigma[A] \in S^{\rho,\lambda}\}$  be the associated  $\Psi$ DO class.

Actually,  $S^{\rho,\lambda}$  fits into the general Hörmander symbol classes (see [31, Chapter XVIII])  $S(m, g)$  with order function  $m(\xi, x) = (1 + |x|^2)^{\rho/2} (1 + |\xi|^2)^{\lambda/2}$  and slowly varying metric  $g_{\xi, x} = (1 + |\xi|^2)^{-1} |d\xi|^2 + (1 + |x|^2)^{-1} |dx|^2$ .

*Second proof of Theorem 3.2.* First, equation (3) and the product formula for  $\Psi$ DOs allows us to compute the symbol of  $\Delta_r^\theta$ :

$$\sigma[\Delta_r^\theta](\xi, x) = \eta^{\mu\nu} \left( \xi_\mu \xi_\nu - 2i\omega_\mu(x - \frac{1}{2}\Theta\xi)\xi_\nu - i\partial_\mu\omega_\nu(x - \frac{1}{2}\Theta\xi) - \omega_\nu \star_\Theta \omega_\mu(x - \frac{1}{2}\Theta\xi) \right) + E(x - \frac{1}{2}\Theta\xi),$$

and because  $\omega_\mu, E \in \mathcal{O}_0(\mathbb{R}^n)$ ,  $\Delta_r^\theta$  lies in  $\Psi^{0,2}$ .

Let  $\{f_N\}_{N \in \mathbb{N}}$  be the family of smooth compactly supported functions defined by  $f_N(x) := \chi_N(x) e^{-x}$ , where  $0 \leq \chi_N \leq 1$ ,  $\chi_N \in \mathcal{C}_c^\infty(\mathbb{R})$  with  $\chi_N(x) = 0$  for  $x \in ]-\infty, -\epsilon] \cup [N, +\infty[$  for a fixed  $\epsilon > 0$  and  $\chi_N(x) = 1$  for  $x \in [0, N - \epsilon]$ . First, [18, Theorem 8.7] yields  $f_N(t\Delta_r^\theta) \in \Psi^{0,-\infty}$ , and basic estimates (see [24, Section 2.4] for details) gives  $L^\Theta(f) \in \Psi^{-\infty,0}$  for all  $f \in \mathcal{S}(\mathbb{R}^n)$ . Then, by [31, Lemma 18.4.3] one obtains  $L^\Theta(f)f_N(t\Delta_r^\theta) \in \Psi^{-\infty,-\infty}$  and its symbol is in  $\mathcal{S}(\mathbb{R}^{2n})$ . Therefore,

$$\begin{aligned} C &:= \sum_{|\alpha|+|\beta| \leq 2n+1} \|\partial_x^\alpha \partial_\xi^\beta \sigma[L^\Theta(f)f_N(t\Delta_r^\theta)]\|_1 \\ &\leq \sum_{|\alpha|+|\beta| \leq 2n+1} C_{\alpha,\beta} \int (1 + |x|^2)^{(-l-|\alpha|)/2} (1 + |\xi|^2)^{(-k-|\beta|)/2} d^n x d^n \xi, \end{aligned}$$

for some  $C_{\alpha,\beta} < \infty$  and all  $l, k \in \mathbb{N}$ , hence  $C < \infty$ . Finally, [18, Theorem 9.4] shows  $L^\Theta(f)f_N(t\Delta_r^\theta)$  is trace-class for all  $N \in \mathbb{N}$ . Looking at the estimates in the proof of [18, Theorem 8.7], one can find constants  $C_{\alpha,\beta}$  independent of  $N$  (because  $e^{-x}$  is rapidly decreasing when  $x \rightarrow +\infty$ , the right support of  $f_N$  plays no role), therefore one obtains that  $L^\Theta(f)f_N(t\Delta_r^\theta)$  is trace-class uniformly in  $N$ .

To finish the proof, it remains to show that  $\text{s-lim } L^\Theta(f)f_N(t\Delta_r^\theta) = L^\Theta(f)e^{-t\Delta_r^\theta}$ , because [17, Proposition 2] will ensure that  $L^\Theta(f)e^{-t\Delta_r^\theta}$  is trace-class for all  $t > 0$ .

Let  $\phi \in \mathcal{H}$  and  $E_\lambda$  be the spectral family of  $\Delta_r^\theta$ , then

$$\begin{aligned} \|(\chi_N(\Delta_r^\theta) - 1)\phi\|_2^2 &= \langle \phi | (\chi_N(\Delta_r^\theta) - 1)^2 \phi \rangle = \int_{\text{Spect}(\Delta_r^\theta)} (\chi_N(\lambda) - 1)^2 d\langle \phi | E_\lambda \phi \rangle \\ &\leq \int_{\text{Spect}(\Delta_r^\theta)} d\langle \phi | E_\lambda \phi \rangle = \langle \phi | \phi \rangle. \end{aligned}$$

Hence by dominated convergence and with  $\phi = e^{-t\Delta_r^\theta} \psi$

$$\begin{aligned} \lim_{N \rightarrow \infty} \|(\chi_N(\Delta_r^\theta) - 1) e^{-t\Delta_r^\theta} \psi\|_2^2 &= \lim_{N \rightarrow \infty} \int_{\text{Spect}(\Delta_r^\theta)} (\chi_N(\lambda) - 1)^2 d\langle e^{-t\Delta_r^\theta} \psi, E_\lambda e^{-t\Delta_r^\theta} \psi \rangle \\ &= \int_{\text{Spect}(\Delta_r^\theta)} \lim_{N \rightarrow \infty} (\chi_N(\lambda) - 1)^2 d\langle e^{-t\Delta_r^\theta} \psi, E_\lambda e^{-t\Delta_r^\theta} \psi \rangle = 0, \end{aligned}$$

where the last equality comes from  $\text{Spect}(\Delta_r^\theta) \subset \mathbb{R}^+$ .  $\square$

We now come to the computation of the exponential of  $\Delta^\Theta$  following a Vassilevich's idea [49].

*Proof of Theorem 2.2.* Let

$$X := 2L^\Theta(\omega_\mu)\partial^\mu + L^\Theta(\partial_\mu\omega^\mu) + L^\Theta(\omega_\mu \star_\Theta \omega^\mu) + L^\Theta(E)$$



$$Y := -\partial_\mu \partial^\mu,$$

so  $\Delta_r^\Theta = Y - X$  and the Baker-Campbell-Hausdorff (BCH) formula

$$e^T e^S = e^{T+S+\frac{1}{2}[T,S]+\frac{1}{12}[T,[T,S]]+\frac{1}{12}[S,[S,T]]-\frac{1}{48}[T,[S,[T,S]]]+\dots},$$

allows to write

$$e^{-t\Delta_r^\Theta} = e^{tX+\frac{1}{2}t^2[X,Y]+\frac{1}{12}t^3[X,[X,Y]]-\frac{1}{6}t^3[Y,[X,Y]]-\frac{1}{48}t^4[X,[Y,[X,Y]]]+\frac{1}{48}t^4[Y,[Y,[X,Y]]]+\dots} e^{-tY}.$$

In order to obtain a power expansion when  $t$  goes to zero, the strategy is to expand the first exponential, to compute the commutators, to reorganize the sequence and finally to write down the explicit symbol of those  $\Psi$ DO's. The trace will be simply taken by integrating them with respect to  $(\xi, x) \in \mathbb{R}^{2n}$ . Actually, the reorganization in homogeneous terms in powers of  $t$  is slightly more elaborate than a simple exponential expansion. All the operators coming from this expansion are of the type  $L^\Theta(g) \partial^\alpha$ ,  $\alpha \in \mathbb{N}^n$ , for some  $g \in \mathcal{B}_\Theta$ . Some terms will give no contributions to the trace since  $\int \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} e^{-t|\xi|^2} d^n \xi = \prod_i \frac{1}{2} (1 + (-1)^{\alpha_i}) \Gamma(\frac{n+1}{2}) t^{-(\alpha_i+1)/2}$  is zero when at least one of the  $\alpha_i$  is odd, and when they are all even,  $|\alpha| = \sum_i \alpha_i = 2l$  is even and we get

$$\int_{\mathbb{R}^n} \xi_{\mu_1} \dots \xi_{\mu_{2l}} e^{-t|\xi|^2} d^n \xi = \left(\frac{\pi}{t}\right)^{n/2} (2t)^{-l} \sum_{\sigma \in S_{2l}} \frac{1}{2^l l!} \delta_{\sigma(\mu_1)\sigma(\mu_2)} \dots \delta_{\sigma(\mu_{2l-1})\sigma(\mu_{2l})},$$

where  $\sigma$  runs over the permutation group  $S_{2l}$  of  $2l$  elements. So, in the reorganization of the power series, we have to keep in mind that  $t^l L^\Theta(g) \partial^\alpha$  is effectively a term of order  $t^{l-\frac{|\alpha|}{2}}$  (independently of the  $(\frac{\pi}{t})^{n/2}$  term.) Moreover, to obtain the asymptotic expansion up to order  $l$  say, we have to use the BCH formula up to order  $2l-1$ . The order of the BCH formula is defined as the number of commutators in the expansion. The term with higher degree derivatives coming from the BCH formula at order  $l$  is

$$[t\partial^2, [t\partial^2, \dots, [t\partial^2, tL^\Theta(g)\partial] \dots]] \propto t^{l+1} L^\Theta(h) \partial^{l+1},$$

for some  $h \in \mathcal{B}_\Theta$ , which yields a term of order  $t^{\frac{l+1}{2}}$ .

Let us show how it works up to order one. We have to use the BCH formula up to order one also:  $e^{-t\Delta_r^\Theta} = e^{tX-tY} = e^{tX+\frac{1}{2}[tX,tY]+\dots} e^{-tY}$ , and

$$\begin{aligned} [tX, tY] &= t^2 [\partial_\nu \partial^\nu, 2L^\Theta(\omega_\mu) \partial^\mu + L^\Theta(\partial_\mu \omega^\mu) + L^\Theta(\omega_\mu \star_\Theta \omega^\mu) + L^\Theta(E)] \\ &= t^2 (2L^\Theta(\partial_\nu \partial^\nu \omega_\mu) \partial^\mu + 4L^\Theta(\partial_\nu \omega_\mu) \partial^\mu \partial^\nu + L^\Theta(\partial_\nu \partial^\nu \partial_\mu \omega^\mu) + 2L^\Theta(\partial_\nu \partial_\mu \omega^\mu) \partial^\nu \\ &\quad + L^\Theta(\partial_\nu \partial^\nu (\omega_\mu \star_\Theta \omega^\mu)) + 2L^\Theta(\partial_\nu (\omega_\mu \star_\Theta \omega^\mu)) \partial^\nu + L^\Theta(\partial_\nu \partial^\nu E) + 2L^\Theta(\partial_\nu E) \partial^\nu) \\ &= 4t^2 L^\Theta(\partial_\nu \omega_\mu) \partial^\mu \partial^\nu + O(t^2), \end{aligned}$$

hence

$$\begin{aligned} L^\Theta(f) e^{-t\Delta_r^\Theta} &= L^\Theta(f) e^{t(2L^\Theta(\omega_\mu) \partial^\mu + L^\Theta(\partial_\mu \omega^\mu) + L^\Theta(\omega_\mu \star_\Theta \omega^\mu) + L^\Theta(E)) + 2t^2 L^\Theta(\partial_\nu \omega_\mu) \partial^\mu \partial^\nu + \dots} e^{t\partial_\mu \partial^\mu} \\ &= L^\Theta(f) \left( 1 + t(2L^\Theta(\omega_\mu) \partial^\mu + L^\Theta(\partial_\mu \omega^\mu) + L^\Theta(\omega_\mu \star_\Theta \omega^\mu) + L^\Theta(E)) \right. \\ &\quad \left. + 2t^2 (L^\Theta(\partial_\nu \omega_\mu) \partial^\mu \partial^\nu + L^\Theta(\omega_\mu \star_\Theta \omega_\nu) \partial^\mu \partial^\nu) + O(t^2) \right) e^{t\partial_\mu \partial^\mu}. \end{aligned}$$

where the last  $t^2$ -term comes from  $e^{tX}$ . So, by (3)

$$\begin{aligned} \sigma \left[ L^\Theta(f) e^{-t\Delta_r^\Theta} \right] (\xi, x) &= \left( f(x - \tfrac{1}{2}\Theta\xi) + t(2f \star_\Theta \omega_\mu(x - \tfrac{1}{2}\Theta\xi)(-i\xi)^\mu \right. \\ &\quad + f \star_\Theta \partial_\mu \omega^\mu(x - \tfrac{1}{2}\Theta\xi) + f \star_\Theta \omega_\mu \star_\Theta \omega^\mu(x - \tfrac{1}{2}\Theta\xi) + f \star_\Theta E(x - \tfrac{1}{2}\Theta\xi)) \\ &\quad + 2t^2(f \star_\Theta \partial_\nu \omega_\mu(x - \tfrac{1}{2}\Theta\xi)(-i\xi)^\mu(-i\xi)^\nu \\ &\quad \left. + f \star_\Theta \omega_\mu \star_\Theta \omega_\nu(x - \tfrac{1}{2}\Theta\xi)(-i\xi)^\mu(-i\xi)^\nu) + O(t^2) \right) e^{-t\xi_\mu \xi^\mu}. \end{aligned}$$

Finally, it remains to integrate  $\sigma \left[ L^\Theta(f) e^{-t\Delta_r^\Theta} \right] (\xi, x)$ . By the translation  $x \rightarrow x + \frac{1}{2}\Theta\xi$ , one obtains:

$$\begin{aligned} \text{Tr} \left( L^\Theta(f) e^{-t\Delta_r^\Theta} \right) &= (2\pi)^{-n} \iint \left( f(x) + t(2f \star_\Theta \omega_\mu(x)(-i\xi)^\mu + f \star_\Theta \partial_\mu \omega^\mu(x) + f \star_\Theta \omega_\mu \star_\Theta \omega^\mu(x) + f \star_\Theta E(x)) \right. \\ &\quad \left. + 2t^2(f \star_\Theta \partial_\nu \omega_\mu(x)(-i\xi)^\mu(-i\xi)^\nu + f \star_\Theta \omega_\mu \star_\Theta \omega_\nu(x)(-i\xi)^\mu(-i\xi)^\nu) \right) e^{-t\xi_\mu \xi^\mu} d^n x d^n \xi \\ &\quad + O(t^{-n/2+2}) \\ &= (4\pi t)^{-\frac{n}{2}} \int f(x) \left( 1 + t(\partial_\mu \omega^\mu(x) + \omega_\mu \star_\Theta \omega^\mu(x) + E(x) - \partial_\mu \omega^\mu(x) - \omega_\mu \star_\Theta \omega^\mu(x)) \right) d^n x \\ &\quad + O(t^{-\frac{n}{2}+2}) \\ &= (4\pi t)^{-\frac{n}{2}} \int f(x) (1 + tE(x)) d^n x + O(t^{-\frac{n}{2}+2}). \end{aligned}$$

The higher order terms can be obtained by similar computations, that is to say, one generically get

$$L^\Theta(f) e^{-t\Delta_r^\Theta} \sim_{t \rightarrow 0} L^\Theta(f) \left( \sum_{l \in \mathbb{N}} t^l \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq l} L^\Theta(g_{\alpha,l}) t^{|\alpha|/2} \partial^\alpha \right) e^{t\partial_\mu \partial^\mu},$$

for some  $g_{\alpha,l} \in \mathcal{B}_\Theta$ , and where we have corrected the powerseries in  $t$  by the order of derivatives, with respect to the previous discussion. Here  $\sim$  means asymptotic expansion with respect to the trace-norm topology:

$$\|L^\Theta(f) e^{-t\Delta_r^\Theta} - L^\Theta(f) \left( \sum_{l \leq N} t^l \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq l} L^\Theta(g_{\alpha,l}) t^{|\alpha|/2} \partial^\alpha \right) e^{t\partial_\mu \partial^\mu}\|_1 = O(t^{N+1}),$$

convergence of the sequence being warranted by Theorem 3.2.

This concludes the proof of Theorem 2.2 since in (11) we get the coefficient  $\text{Tr}(1_{\mathbb{C}^{2m}})$ .  $\square$

*Remark 3.4.* This systematic computation also yields that the other coefficients  $\tilde{a}_{2l}$ ,  $l > 3$  have the same canonical form, that is Moyal products replace pointwise ones everywhere.

### 3.2 Heat kernel expansion for NC tori in Moyal (re)presentation

Let  $\mathcal{A}_\Theta$  be the smooth algebra of Schwartz (rapidly decreasing) linear combination of the plane waves  $\{e^{ik \cdot x}\}_{k \in \mathbb{Z}^n}$  endowed with Moyal product:

$$\mathcal{A}_\Theta = \left( \left\{ \sum_{k \in \mathbb{Z}^n} c_k e^{ik \cdot x} : (c_k) \in \mathcal{S}(\mathbb{Z}^n) \right\}, \star_\Theta \right).$$

$\mathcal{A}_\Theta$  closes to an algebra and represents the NC  $n$ -tori:

$$e^{ik \cdot x} \star_\Theta e^{iq \cdot x} = e^{-ik \cdot \Theta q} e^{iq \cdot x} \star_\Theta e^{ik \cdot x}, \quad (13)$$

this canonical commutation relation of the NC  $n$ -tori coming from the straightforward computation (here Fourier modes are viewed as tempered distributions):

$$e^{ik \cdot x} \star_\Theta e^{iq \cdot x} = (2\pi)^{-n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\xi \cdot (x-y)} e^{ik \cdot (x - \frac{1}{2}\Theta\xi)} e^{iq \cdot y} d^n\xi d^n y = e^{-i\frac{1}{2}k \cdot \Theta q} e^{i(k+q) \cdot x}.$$

One can build a unital spectral triple associated to this algebra [11, 28], with  $\mathcal{H} = L^2(\mathbb{T}^n) \otimes \mathbb{C}^{2^m}$  the squared integrable sections of the trivial spinor bundle over  $\mathbb{T}^n$ , and  $\mathcal{D} = \not{D}$  the flat Dirac operator.  $\mathcal{A}_\Theta$  is again represented on bounded operators by the left regular representation  $\pi^\Theta(a)\psi = L^\Theta(a) \otimes 1_{2^m} \psi = a \star_\Theta \psi$ , for  $a \in \mathcal{A}_\Theta, \psi \in \mathcal{H}$ . Actually, this construction is equivalent to the GNS representation associated to the state given by the canonical trace  $\tau$  of  $\mathcal{A}_\Theta$ : when  $a(x) = \sum_{k \in \mathbb{Z}^n} c_k e^{ik \cdot x} \in \mathcal{A}_\Theta$ ,

$$\tau(a) := c_0 = \int_{\mathbb{T}^n} a(x) d^n x.$$

Let again  $\Delta^\Theta$  be the noncommutative generalized Laplacian defined in (10) acting now on  $\mathcal{H} := L^2(\mathbb{T}^n) \otimes \mathbb{C}^{2^m}$ , where  $\omega_\mu^* = -\omega_\mu$  and  $E \leq 0$  are in  $\mathcal{A}_\Theta$ .

We will first show that in the NC-tori cases,  $e^{-t\Delta^\Theta}$  is trace-class for  $t \in \mathbb{R}_+^*$  is a direct consequence of the compactness of  $\mathcal{R}_\Delta^\Theta(z) := (\Delta^\Theta - z)^{-1}$ . Then, thanks to the previous section, it will be straightforward to show that its trace has a small- $t$  asymptotic expansion (11) where the local invariants  $\tilde{a}_l$  are the same as in the Moyal plane case, but with  $f = 1$ .

**Theorem 3.5.** *Let  $\Delta^\Theta$  be as in (10), then  $e^{-t\Delta^\Theta}$  is trace-class for all  $t \in \mathbb{R}_+^*$ .*

*Proof.* The proof is simpler than for the Moyal plane. Clearly,  $\mathcal{R}_\Delta^\Theta(z) \in \mathcal{L}^p(\mathcal{H})$ ,  $p > n$ , and so  $\mathcal{R}_{\Delta^\Theta}(z) \in \mathcal{L}^{p/2}(\mathcal{H})$  (use the same trick as in the proof of Theorem 3.2). Then Theorem 3.1 yields  $e^{-t\Delta^\Theta} = (e^{-t\frac{2}{p}\Delta^\Theta})^{\frac{p}{2}} \in \mathcal{L}^1(\mathcal{H})$ .  $\square$

For the computation of the small- $t$  expansion, because all algebraic properties (mainly Leibniz rule) used in the previous section work as well as for the tori, we also obtain

$$e^{-t\Delta^\Theta} \sim_{t \rightarrow 0} \sum_{l \in \mathbb{N}} t^l \left( \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq l} L^\Theta(g_{\alpha,l}) t^{|\alpha|/2} \partial^\alpha \right) e^{t\partial_\mu \partial^\mu},$$

where  $g_{\alpha,l} \in \mathcal{A}_\Theta$  are the same as for the Moyal plane case except that now  $f = 1$ . So the trace has the small- $t$  expansion:

$$\begin{aligned} \text{Tr} \left( e^{-t\Delta^\Theta} \right) &= 2^m (2\pi)^{-n} \int_{\mathbb{T}^n} d^n x \int_{\mathbb{R}^n} \sigma \left[ e^{-t\Delta^\Theta} \right] (\xi, x) d^n \xi \\ &\sim_{t \rightarrow 0} \frac{2^m}{(2\pi)^n} \int_{\mathbb{T}^n} d^n x \int_{\mathbb{R}^n} \sum_{l \in \mathbb{N}} t^l \left( \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq l} g_{\alpha,l}(x - \frac{1}{2}\Theta\xi) t^{|\alpha|/2} (-i\xi)^\alpha \right) e^{-t|\xi|^2} d^n \xi. \end{aligned}$$

Now, expanding  $g_{\alpha,l}$  in Fourier modes and using

$$\int_{\mathbb{T}^n} e^{ik(x - \frac{1}{2}\Theta\xi)} d^n x = e^{-ik\frac{1}{2}\Theta\xi} \delta_{k,0} = \delta_{k,0},$$

we directly obtain the following result:

**Theorem 3.6.** *Let  $\Delta^\Theta$  be as in (10), then*

$$\text{Tr} \left( e^{-t\Delta^\Theta} \right) \sim_{t \rightarrow 0} 2^m \left( \frac{1}{4\pi t} \right)^{n/2} \sum_{l \in \mathbb{N}} t^l \int_{\mathbb{T}^n} \tilde{a}_{2l}(x) d^n x,$$

where the  $\tilde{a}_{2l}(x)$  are given in Theorem 2.2.

## 4 The spectral action

### 4.1 Spectral action for nonunital spectral triples

For a unital spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ , Chamseddine and Connes [5, 6] proposed a definition of a physical action which depends only on the spectrum of the covariant  $\mathcal{D}$ -operator (the spectral action principle):

$$S(\mathcal{D}, A) := \text{Tr}(\phi(\mathcal{D}_A^2/\Lambda^2)), \quad (14)$$

where  $\mathcal{D}_A$  is the covariant "Dirac" operator  $\mathcal{D}_A := \mathcal{D} + A + \epsilon J A J^{-1}$ ,  $A$  is a universal represented 1-form  $A \in \tilde{\pi}(\Omega^1 \mathcal{A})$ ,  $\tilde{\pi}$  being the lifted representation on the whole differential algebra  $\Omega^\bullet \mathcal{A}$  ( $\tilde{\pi}(a_0 \delta a_1 \cdots \delta a_p) := \pi(a_0)[\mathcal{D}, \pi(a_1)] \cdots [\mathcal{D}, \pi(a_p)]$ ,  $a_i \in \mathcal{A}$ ,  $i = 1, \dots, p$ ),  $J$  is the real structure of the triple (the charge conjugation for spinors in the commutative case),  $\phi$  a suitable cut-off function,  $\Lambda$  a mass scale and  $\epsilon \in \{+1, -1\}$  depending upon the dimension. Any positive smooth function  $\phi$  mimicking the step function  $\chi_{[0,1]}$  was initially used in [5, 6] and in [21], sufficient conditions on  $\phi$  have been detailed. Since in the unital case,  $\mathcal{D}$  has compact resolvent and likewise for the perturbed  $\mathcal{D}_A$  by Theorem 3.5,  $\phi(\mathcal{D}_A^2/\Lambda^2)$  is trace-class as long as  $\phi$  decreases fast enough; for instance  $r^{n-1}\phi(r^2) \in L^1(\mathbb{R}^+)$  is a sufficient condition for a spectral triple with spectral dimension equal to  $n$ .

Let us be more explicit about the covariant "Dirac" operator  $\mathcal{D}_A$ . The starting point is the analogy between the invariance group of a gauge theory on a Riemannian manifold  $M$  coupled with general relativity,  $G = U \rtimes \text{Diff}(M)$  and the group of automorphism of an algebra  $\mathcal{A}$  which splits into its inner and outer part  $\text{Aut}(\mathcal{A}) = \text{Int}(\mathcal{A}) \rtimes \text{Out}(\mathcal{A})$ , with the following exact (group) sequence:

$$\begin{aligned} 1 &\rightarrow U \rightarrow G \rightarrow \text{Diff}(M) \rightarrow 1, \\ 1 &\rightarrow \text{Int}(\mathcal{A}) \rightarrow \text{Aut}(\mathcal{A}) \rightarrow \text{Out}(\mathcal{A}) \rightarrow 1. \end{aligned}$$

In particular, if we choose  $\mathcal{A} = \mathcal{C}^\infty(M, M_n(\mathbb{C})) \cong \mathcal{C}^\infty(M) \otimes M_n(\mathbb{C})$ ,  $n > 1$ , the two constructions coincide:  $\text{Out}(\mathcal{A}) = \text{Diff}(M)$ ,  $\text{Int}(\mathcal{A}) = \mathcal{C}^\infty(M, SU_n/\mathbb{Z}_n)$ . The natural invariance group for an action defined on a spectral triple must be the automorphism group of the algebra. In order to retrieve a gauge theory with spin matter when  $\mathcal{A}$  is almost commutative that is  $\mathcal{A} = \mathcal{C}^\infty(M) \otimes A_F$  (where  $A_F$  is a finite algebra such as  $\mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$  for the standard model of particle physics [2, 5, 6]), we must represent  $\text{Aut}(\mathcal{A})$  in the fermionic Hilbert space  $\mathcal{H}$ . In particular, we have to lift  $\text{Int}(\mathcal{A})$  to the unitary group  $\mathcal{U}(\mathcal{H})$  of the bounded operators on  $\mathcal{H}$ :

$$\mathcal{U}(\mathcal{A}) \ni u \mapsto \sigma(u) = \pi(u) J \pi(u) J^{-1} \in \mathcal{U}(\mathcal{H}).$$

For NC tori, Moyal planes and some almost-commutative geometries, this is the adjoint representation:  $\pi(u) J \pi(u) J^{-1} \psi = u \star_\Theta \psi \star_\Theta u^*$ ,  $\psi \in \mathcal{H}$ . Under this transformation,  $\mathcal{D}$  transforms as

$$\mathcal{D} \rightarrow \sigma(u) \mathcal{D} \sigma(u)^{-1} = \mathcal{D} + \pi(u) [\mathcal{D}, \pi(u^*)] + \epsilon J \pi(u) [\mathcal{D}, \pi(u^*)] J^{-1}, \quad (15)$$

where  $\epsilon$  comes from commutation relations  $\mathcal{D} J = \epsilon J \mathcal{D}$ ,  $\epsilon \in \{+1, -1\}$  (see [10, 28] for a table of signs)<sup>1</sup>. Hence  $\mathcal{D}_A \rightarrow \mathcal{D}_{A'}$  with  $A' = \pi(u) A \pi(u^*) + \pi(u) [\mathcal{D}, \pi(u^*)]$  transforms covariantly.

For almost commutative geometry  $\mathcal{C}^\infty(M) \otimes A_F$ , in particular for the standard model, with  $\mathcal{D} = \not{D} \otimes 1_{\mathcal{H}_F}$  and the curved Dirac operator  $\not{D} = -ie^\mu_a \gamma^a (\partial_\mu + \omega_\mu)$ ,  $\omega$  being the spin connection on  $M$ ,  $S(\mathcal{D}, A)$  is asymptotically computable by heat kernel techniques. We may note that  $\not{D}_A^2$  can be written as a generalized Laplacian:  $\not{D}_A^2 = P$  with  $P = -(g^{\mu\nu} (\partial_\mu + \omega_\mu)(\partial_\nu + \omega_\nu) + E)$  where  $g^{\mu\nu}$  is the metric tensor, now  $\omega_\mu$  is a connection containing spin and Yang-Mills part

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<sup>1</sup>The sign  $\epsilon$  in equation (15) is actually wrong in most of the literature, however the computations linked with physics models are unaffected because  $\epsilon = 1$  in the zero and four dimensional cases.

and  $E$  is an endomorphism of the fiber bundle, on whose sections  $P$  acts. One can formally show [5,6], expanding  $\phi$  in Taylor series, that  $S(\mathcal{D}, A)$  is linked to the Seeley–DeWitt coefficients  $a_k(P, x)$  of the trace of the heat operator on a  $n$ -dimensional manifold

$$\mathrm{Tr}(e^{-tP}) \sim_{t \rightarrow 0} (4\pi)^{-n/2} \sum_{l \in \mathbb{N}} t^{(l-n)/2} \int_M a_l(P, x) \, \mathrm{dvol}(x). \quad (16)$$

where  $\mathrm{dvol}(x)$  is the Riemannian volume form, by the relation between zeta function and trace of the heat operator [25]:

$$\zeta_P(s) := \mathrm{Tr}(P^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}(e^{-tP}) \, dt. \quad (17)$$

On a manifold without boundary  $a_l(P, x) = 0$ ,  $l$  odd, therefore in the four dimensional case, this yields:

$$S(\mathcal{D} \otimes 1_{A_F}, A) = (4\pi)^{-2} \sum_{l=0}^2 \Lambda^{4-2l} \phi_{2l} \int_M a_{2l}(P, x) \, \mathrm{dvol}(x) + O(\Lambda^{-2}), \quad (18)$$

where

$$\phi_0 = \int_0^\infty \phi(t) \, t \, dt, \quad \phi_2 = \int_0^\infty \phi(t) \, dt, \quad \phi_{2(2l+2)} = (-1)^l \phi^{(l)}(0), \quad l \geq 0. \quad (19)$$

A less formal, derivation of this relation with precise constraints on  $\phi$  can be found in [21, 40]. For  $M$  still four dimensional and  $A_F = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$ , the spectral action yields a unification of the Einstein plus Weyl gravity and the standard model including the Higgs sector and its spontaneous symmetry breaking (see [2, 5, 6]). There is no restriction for an arbitrary dimension but the coefficients (19) will be slightly different, as we will see below for the Moyal plane.

*Remark 4.1.* The relation (17), links also the Dixmier trace with the heat kernel expansion, and therefore the Connes–Lott action (8) with the spectral action as explained in section 4.3.

For the nonunital case, since  $\mathcal{D}$  has no longer a compact resolvent, we invoke a *spatial regularization*  $\rho$  to define the spectral action. Like the energy regularization  $\phi$ ,  $\rho$  is a positive function rapidly decreasing, in the almost commutative case, and must be generically an element of the algebra  $\mathcal{A}$ .

**Definition 4.2.** For a nonunital spectral triple  $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{H}, \mathcal{D})$  of spectral dimension  $n$ , the *spectral action* is

$$S(\mathcal{D}, A, \rho) := \mathrm{Tr}_{\mathcal{H}}(\pi(\rho) \phi(\mathcal{D}_A^2 / \Lambda^2)), \quad (20)$$

where as in the unital case,  $\mathcal{D}_A = \mathcal{D} + A + \epsilon J A J^{-1}$  ( $\epsilon \in \{+1, -1\}$  depending on  $n$ ), and  $A \in \Omega_{\mathcal{D}}^1(\tilde{\mathcal{A}})$  is now a represented selfadjoint 1-form of the unitized algebra:  $A = \sum_{i \in I} \pi(b_0^i) [\mathcal{D}, \pi(b_1^i)]$ , for  $I$  a finite set,  $b_0^i, b_1^i \in \tilde{\mathcal{A}}$ ,  $0 \leq \rho \in \mathcal{A}$  and moreover  $0 \leq \phi, \Lambda$  are as in the unital case.

*Remark 4.3.* i) This definition gives more importance to the choice of the unitization. The 1-form  $A$  is now constructed from  $\tilde{\mathcal{A}}$ , and all the symmetry considerations discussed previously occur now for the unitized algebra. This is important because unitaries in the algebra are necessary to express gauge invariance:  $S(\mathcal{D}, A, \rho)$  is gauge invariant that is invariant under the lifted inner automorphism implemented by the unitary operator  $\pi(u) J \pi(u) J^{-1}$  and now the regularization  $\rho$  transforms also:

$$\begin{cases} A & \rightarrow u A u^* + u [\mathcal{D}, u^*], \\ \rho & \rightarrow u \rho u^*. \end{cases}$$

ii) The positivity of  $\rho$  and  $\phi$  is necessary in order to get a positive action.

iii) Other regularizations are possible. For instance,  $\phi(\mathcal{D}_A^2 \pi(\rho)^{-1})$  where  $f \in \mathcal{S}$  is a strictly

positive function also give rise to trace-class operators for Moyal planes, but the asymptotic expansion is still unmanageable.

Let us show how it works for an almost commutative geometry associated with a boundaryless noncompact smooth manifold  $M$ . In this case, we still work with  $\mathcal{A} = \mathcal{C}_c^\infty(M) \otimes A_F$ . The operator  $\rho e^{-tP}$ ,  $t \in \mathbb{R}_+^*$ , is trace-class, for  $\rho \in \mathcal{C}_c^\infty(M)$  viewed as a pointwise multiplication operator and  $P$  being a generalized Laplacian ( $\Psi$ DO operator of order two with metric tensor as coefficient of the leading symbol). In this case the formula (17) has an analogue:

$$\zeta_{\rho,P}(s) := \text{Tr}(\rho P^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(\rho e^{-tP}) dt, \quad (21)$$

one obtains, for  $P = (\mathcal{D} + A + \epsilon JAJ^{-1})^2$ ,

$$S(\mathcal{D} \otimes 1_{A_F}, A) = (4\pi)^{-n/2} \sum_{l=0}^m \Lambda^{n-2l} \phi_{2l} \int_M a_{2l}(P, x) \rho(x) \text{dvol}(x) + O(\Lambda^{n-2(m+1)}),$$

where  $a_{2l}$  are still the Seeley–DeWitt coefficients which are now only locally integrable while  $a_{2l}(P, x)\rho(x)$  are globally integrable (see [50]). The coefficients  $\phi_{2l}$  have the form (19) in the four dimensional case, and their values in any dimension  $n$  is now computed for Moyal planes.

## 4.2 The case of the Moyal plane

Actually, the relation (21) is quite general, that is for any bounded operator  $S$  and any operator  $T$  such that  $ST^{-s}$  is trace-class, we have

$$\zeta_{S,T}(s) := \text{Tr}(ST^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(S e^{-tT}) dt. \quad (22)$$

With this relation and the result of section 3, one can derive the spectral action for Moyal planes. However in order to obtain more directly the form of the coefficients  $\phi_{2k}$  in any dimension, we will derive it by Laplace transform techniques such as in [40] (see [52] for details on Laplace transform). We assume that the function  $\phi$  has the following property:

$$\phi \in \mathcal{C}^\infty(\mathbb{R}^+) \text{ is the Laplace transform of } \hat{\psi} \in \mathcal{S}(\mathbb{R}^+) := \{g \in \mathcal{S} : g(x) = 0, x \leq 0\} \quad (23)$$

Thus, any function with this property has necessarily an analytic extension on the right complex plane and is a Laplace transform. Consequently, any  $m$ -differentiable function  $\psi$  such that  $\psi^{(m)} = \phi$  is the Laplace transform of a function  $\hat{\psi}$  and by differentiation, it satisfies

$$\phi(z) = \psi^{(m)}(z) = (-1)^m \int_0^\infty e^{-sz} s^m \hat{\psi}(s) ds, \quad \Re z > 0.$$

With  $\Delta^\Theta$  defined in (10), using  $\phi(\Delta_r^\Theta) = (-1)^m \int_0^\infty e^{-s\Delta_r^\Theta} s^m \hat{\psi}(s) ds$  and the positivity of  $\rho = g^* \star_\Theta g$ ,  $g \in \mathcal{B}_\Theta$ , we get

$$\text{Tr}(L^\Theta(\rho) \phi(\Delta_r^\Theta/\Lambda^2)) = (-1)^m \text{Tr}\left(L^\Theta(g) \int_0^\infty e^{-t\Delta_r^\Theta/\Lambda^2} t^m \hat{\psi}(t) dt L^\Theta(g^*)\right).$$

Let  $\{\Phi_p\}_{p \in \mathbb{N}}$  be any orthonormal basis of  $\mathcal{H}_r$  and let  $0 \leq B_t := L^\Theta(g) e^{-t\Delta_r^\Theta/\Lambda^2} L^\Theta(g^*)$ , then

$$\begin{aligned} \text{Tr}(L^\Theta(\rho) \phi(\Delta_r^\Theta/\Lambda^2)) &= \lim_{N \rightarrow \infty} \int_0^\infty \sum_{p \leq N} \langle \Phi_p, B_t \Phi_p \rangle t^m \hat{\psi}(t) dt \\ &\leq \lim_{N \rightarrow \infty} \int_0^\infty \|B_t\|_1 t^m \hat{\psi}(t) dt = \int_0^\infty \|B_t\|_1 t^m \hat{\psi}(t) dt. \end{aligned}$$

Let us estimate  $\|B_t\|_1$ . For  $t > \epsilon$  with a fixed arbitrary small  $\epsilon$ , we have:

$$\|B_t\|_1 = \|L^\Theta(g)e^{-t\Delta_r^\Theta/2\Lambda^2}\|_2^2 \leq \|L^\Theta(g)e^{-\epsilon\Delta_r^\Theta/2\Lambda^2}\|_2^2 \|e^{-(t-\epsilon)\Delta_r^\Theta/2\Lambda^2}\|.$$

But,  $(t - \epsilon)\Delta_r^\Theta$  being positive, we have  $\|e^{-(t-\epsilon)\Delta_r^\Theta/2\Lambda^2}\| \leq 1$ . Hence for  $t > \epsilon$ ,  $\|B_t\|_1 \leq C$  uniformly in  $t$ . For  $t \leq \epsilon$ , our previous computation shows that  $\|B_t\|_1 = O(t^{-n/2})$ . Hence  $\text{Tr} \left( \int_0^\infty B_t t^m \hat{\psi}(t) dt \right) < \infty$ , so by dominated convergence one obtains:

$$\begin{aligned} & \text{Tr} (L^\Theta(\rho) \phi(\Delta_r^\Theta/\Lambda^2)) \\ &= (-1)^m \int_0^\infty \text{Tr} \left( L^\Theta(\rho) e^{-t\Delta_r^\Theta/\Lambda^2} \right) t^m \hat{\psi}(t) dt \\ &= (-1)^m (4\pi)^{-n/2} \int_0^\infty \sum_{l=0}^m \Lambda^{n-2l} t^{m+l-n/2} \hat{\psi}(t) dt \int_{\mathbb{R}^n} \rho \star_\Theta \tilde{a}_{2l}(x) d^n x + O(\Lambda^{n-2(m+1)}) \\ &= (4\pi)^{-n/2} \sum_{l=0}^m \Lambda^{n-2l} \phi_{2l} \int_{\mathbb{R}^n} \rho \star_\Theta \tilde{a}_{2l}(x) d^n x + O(\Lambda^{n-2(m+1)}), \end{aligned}$$

where  $\phi_{2l}$  is now defined by

$$\phi_{2l} := (-1)^m \int_0^\infty t^{m+l-n/2} \hat{\psi}(t) dt. \quad (24)$$

When  $n = 2m$  is even,  $\phi_{2l}$  has the more familiar form of (19):

$$\phi_{2l} = \begin{cases} \frac{1}{\Gamma(m-l)} \int_0^\infty \phi(t) t^{m-1-l} dt, & \text{for } l = 0, \dots, m-1, \\ (-1)^l \phi^{(l-m)}(0), & \text{for } l = m, \dots, n. \end{cases} \quad (25)$$

For  $n$  odd, the coefficients  $\phi_{2l}$  have less explicit forms because they invoke fractional derivatives of  $\phi$ , so in this case, it is better to stick to definition (24).

Let us summarize:

**Theorem 4.4.** *Let  $\rho \in \mathcal{S}(\mathbb{R}^n)$ ,  $A = -iL^\Theta(A_\mu) \otimes \gamma^\mu$ ,  $A_\mu^* = -A_\mu \in \mathcal{O}_0(\mathbb{R}^n)$ ,  $\phi \in \mathcal{C}^\infty(\mathbb{R}^+)$  be a positive function satisfying condition (23) and  $\not{D}_A = \not{D} + A$ . Then  $L^\Theta(\rho) \phi(\not{D}_A^2/\Lambda^2)$  is trace-class. Moreover, the following expansion of the spectral action holds:*

$$S(\not{D}, A, \rho) = 2^m (4\pi)^{-n/2} \sum_{l=0}^m \Lambda^{n-2l} \phi_{2l} \int_{\mathbb{R}^n} \rho(x) \tilde{a}_{2l}(x) d^n x + O(\Lambda^{n-2(m+1)}),$$

where the  $\phi_{2l}$  are defined in (24) or (25) depending on the dimension and the  $\tilde{a}_{2l}(x)$  are given in Theorem 2.2 with the following replacement in (10):

$$\begin{cases} L^\Theta(\omega_\mu) & \rightarrow L^\Theta(A_\mu), \\ L^\Theta(E) \otimes 1_{2^m} & \rightarrow (L^\Theta(\partial_\mu A_\nu) + L^\Theta(A_\mu \star_\Theta A_\nu)) \otimes \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu). \end{cases}$$

Moreover, all terms in  $\tilde{a}_{2l}$  linear in  $E$  are zero.

*Proof.* This follows from  $\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ , so all linear terms in  $E$  are of zero trace.  $\square$

**Remark 4.5.** When the Dirac operator is symmetrized,  $\mathcal{D}_A = \mathcal{D} + A + \epsilon J A J^{-1}$ , one has to replace  $L^\Theta(A_\mu)$  by  $L^\Theta(A_\mu) - R^\Theta(A_\mu)$  since  $\epsilon J (L^\Theta(A_\mu) \otimes \gamma^\mu) J^{-1} = R^\Theta(A_\mu^*) \otimes \gamma^\mu$ . So, the behaviour in  $t$  of different terms like  $\text{Tr} (L^\Theta(f) R^\Theta(g) \partial^\alpha e^{t\partial_\mu \partial^\mu})$  has to be computed. Since  $\sigma[L^\Theta(f) R^\Theta(g)](\xi, x) = f(x - \frac{1}{2}\theta\xi) g(x + \frac{1}{2}\theta\xi)$ , the translation invariance  $x \rightarrow x + \frac{1}{2}\theta\xi$  crucially used in the proof of Theorem 2.2 now fails. This point is related to the UV/IR mixings and has to be clarified.

### 4.3 Connes–Lott versus Chamseddine–Connes actions

In order to compare this result with the Connes–Lott action computation of the four dimensional Moyal plane [23], up to negative order terms with respect to the mass scale  $\Lambda$ , we obtain:

$$S(\not{\partial}, A, \rho) = \frac{1}{4\pi^2} \left( \Lambda^4 \phi_0 \int_{\mathbb{R}^4} \rho(x) d^4x + \frac{\phi(0)}{6} \int_{\mathbb{R}^4} \rho(x) F^{\mu\nu} \star_{\Theta} F_{\mu\nu}(x) d^4x \right) + O(\Lambda^{-2}),$$

where  $F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu]_{\star_{\Theta}}$ .

If we choose the characteristic function  $\rho = \chi_V$  of a bounded subset  $V \subset \mathbb{R}^4$  ( $\chi_V \notin \mathcal{S}(\mathbb{R}^4)$ ), property (6) yields:

$$S(\not{\partial}, A, \chi_V) = \frac{1}{4\pi^2} \left( \Lambda^4 \phi_0 \int_V d^4x + \frac{\phi(0)}{6} \int_V F^{\mu\nu} \star_{\Theta} F_{\mu\nu}(x) d^4x \right) + O(\Lambda^{-2}), \quad (26)$$

which, modulo a cosmological term, is the spatially localized noncommutative Yang–Mills action. This expression has to be compared with the four dimensional Connes–Lott one (8) for  $\Theta$  symplectic, hence  $\star_{\Theta} = \star_{\theta}$ :

$$YM(\alpha) = -\frac{1}{4g^2} \int F^{\mu\nu} \star_{\theta} F_{\mu\nu}(x) d^4x.$$

This action is slightly different from (26), because property (6), together with the absence of  $\rho$  gives

$$YM(\alpha) = -\frac{1}{4g^2} \int F^{\mu\nu}(x) F_{\mu\nu}(x) d^4x.$$

For the noncommutative tori, we have also a similar result, with the spectral action in the unital case ( $\rho = 1$ ):

$$S(\not{\partial}, A) \sim_{\Lambda \rightarrow \infty} 2^m (4\pi)^{-n/2} \sum_{k \in \mathbb{N}} \Lambda^{n-2k} \phi_{2k} \int_{\mathbb{T}^n} \tilde{a}_{2k}(x) d^n x,$$

which also yields for  $n = 4$ :

$$S(\not{\partial}, A) = \frac{1}{4\pi^2} \left( \Lambda^4 \phi_0 + \frac{\phi(0)}{6} \int_{\mathbb{T}^4} F^{\mu\nu} \star_{\Theta} F_{\mu\nu}(x) d^4x \right) + O(\Lambda^{-2}).$$

### 4.4 Towards a gravitational degree of freedom

One can ask about adding gravitational degrees of freedom for Moyal planes or NC tori. For instance, the results of Section 3 also work with a non constant metric  $g^{\mu\nu}(x)$ . More precisely, Theorem 3.2 is still true if we replace in (10),  $\Delta^{\Theta}$  by the square of

$$-ie_a^{\mu} (\partial_{\mu} + \omega_{\mu} + L^{\Theta}(A_{\mu})) \otimes \gamma^a,$$

where  $e_a^{\mu}$  and  $\omega_{\mu}$  are bounded functions. Here, the poinwise and Moyal products are mixed, but in this case, computation of the trace of its regularized semigroup can be done, at least in principle, with the same techniques but it will be highly less easy.

However, this construction is meaningless from a spectral triple point of view. A non flat Dirac operator over  $\mathbb{R}^n$ ,  $\not{D} = -ie_a^{\mu}(x)\gamma^a(\partial_{\mu} + \omega_{\mu}(x))$ ,  $\omega_{\mu}$  being the spin connection, will violate most of the axioms describing spectral triples, for instance

$$[\not{D}, \pi^{\Theta}(f)] = -i\gamma^a ([e_a^{\mu} \omega_{\mu}, \pi^{\Theta}(f)] + e_a^{\mu} \pi^{\Theta}(\partial_{\mu} f) + [e_a^{\mu}, \pi^{\Theta}(f)] \partial_{\mu}).$$



So, for  $f \in \mathcal{B}_\Theta$ ,  $[\mathcal{D}, \pi^\Theta(f)]$  can be extended to a bounded operator only if  $[e_a^\mu, \pi^\Theta(f)] = 0$ . This condition can be satisfied for instance by a  $n$ -dimensional Riemannian manifold  $(M, g)$  endowed with an isometric action of  $\mathbb{R}^l$ ,  $l \geq 2$  (periodic or not). This is actually the Connes–Landi isospectral deformations [13, 14]. Those cases, admit nontrivial fluctuations of the metric (in some sense for the untwisted directions). Since the only invariant metric on  $\mathbb{R}^n$  or  $\mathbb{T}^n$  by the natural action of  $\mathbb{R}^n$  is the flat one and this is the geometrical obstruction to deal with nonflat Moyal planes (see [4]).

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